Fundamentals of the nonlinear theory of photorefractive subharmonics

B. I. Sturman, Marìa Aguilar, and F. Agullò-Lòpez
Universidad Autonoma, 28049 Madrid, Spain

K. H. Ringhofer*
Fachbereich Physik der Universität, D-49069 Osnabrück, Germany

(Received 29 August 1996)

We formulate the fundamentals of the nonlinear theory of low-frequency space-charge waves in semi-insulating photorefractive crystals. This includes an analysis of dispersion relations for the waves, and of their parametric excitation by a running light pattern, a description of various stationary states (split and unsplit subharmonics) beyond the threshold of the parametric instability, and a study of the stability of those nonlinear steady states against small perturbations. Nonlinear eigenfrequency and mutual frequency shifts for strong waves and renormalization of the coupling coefficients for weak waves are important elements of the theory. An investigation of the stability of the subharmonics also incorporates their phase relations as well as certain special features of space-charge waves. One of the consequences of the theory is the modulational instability of the main subharmonic, characterized by doubling of the period of the primary light pattern. Finally we discuss further development and applications of the theory. [S1063-651X(97)10604-3]

PACS number(s): 42.40.Pa, 42.65.Hw, 42.70.Nq

I. INTRODUCTION

The problem of how to describe the equilibrium state of a physical system governed by nonlinear evolutional equations is of a very general nature. It arises in hydrodynamics [1,2], plasma physics [3,4], nonlinear optics [5,6], and in many other fields of physics. An analysis of the stability of the stationary solutions often reveals the details of the transition from a deterministic (laminar) behavior of a physical system to a chaotic (turbulent) one with increasing strength of the nonlinearity [1,7].

Investigations of the so-called weakly nonlinear wave interactions figure prominently in the above subject [4,8,9]. The existence of a small physical parameter—the ratio of the damping $\gamma_{\vec{k}}$ of the wave with wave vector $\vec{k}$ to its frequency, $\omega_{\vec{k}}$—enables one to advance greatly in the description and understanding of nonlinear wave phenomena even without numerical simulations, if the wave amplitudes are sufficiently small. The simple idea that the waves remain quasimonocromatic and quasiplane, so that the concept of resonance (linear or nonlinear) still holds true, lies at the heart of such studies.

The scope of weakly nonlinear wave phenomena is quite large: among them are dynamic effects with narrow wave packets and kinetic effects involving wide wave spectra. The latter case is known as ''weak wave turbulence'' [4]. Substantial progress in the field of weakly nonlinear wave phenomena has been achieved in plasma physics [8] and in ferromagnetism [9].

The experience of numerous studies shows that the behavior of waves above the threshold of their nonlinear excitation is highly sensitive to the special features of the physical system: to the method of pumping, to the dispersion law $\omega_{\vec{k}}$, to the dependence of the nonlinear coupling coefficients on the wave vector, etc. A universal description of the above-threshold behavior of nonlinear waves is hardly possible.

In the present paper we are dealing with the above-threshold regimes and their stability as applied to space-charge waves in photosensitive dielectrics. Although such waves were predicted more than 20 years ago [10], real interest in them arose quite recently during studies of the nonlinear photorefractive phenomena in crystals of the sillenite family, $Bi_{12}SiO_{20}$, $Bi_{12}TiO_{20}$, and $Bi_{12}GeO_{20}$.

A brief history of space-charge waves is as follows. It was found in 1988 [11] that the space-charge field $E_{sc}(\vec{x})$ created in a $Bi_{12}SiO_{20}$ crystal by a moving light pattern under certain conditions looses the periodicity of the external exposure. The Fourier spectrum of $E_{sc}(\vec{x})$, apart from the fundamental spatial frequency of the interference pattern $\vec{K}$, and the higher harmonics $2\vec{K}$, $3\vec{K}$, ... , also included the fractional frequencies $\vec{K}/2$, $\vec{K}/3$, and $\vec{K}/4$ (spatial subharmonics). Further experiments revealed that the subharmonics can also be excited in other crystals of the sillenite family [12–14]. Furthermore it has been shown that doubling and tripling of the spatial period are also possible in the presence of a standing light pattern and an external ac field [13,15]. Later it was detected that the first subharmonic may split [16,17]: instead of a single spatial frequency $\vec{K}/2$, two frequencies $\vec{k}_1$ and $\vec{k}_2$ near $\vec{K}/2$ were observed in the Fourier spectrum; their sum was equal to $\vec{K}$. The splitting can be parallel (longitudinal) or perpendicular (transverse) to $\vec{K}$. It should be noted that the Fourier spectrum of the space-charge field in photorefractive crystals is easily visualized on a screen by means of light diffraction.

Theoretical investigations have shown [18–20] that the generation of photorefractive subharmonics (split or unsplit) is due to the parametric instability against excitation of weakly damped low-frequency space-charge waves, which

*Electronic address: ringhofe@mail.rz.uni-osnabrueck.de

exist in electrically biased sillenite crystals. Additional experiments [21,22] directly confirmed this conclusion. The theory developed in Refs. [18–20] enabled one to explain a fair amount of experimental data on subharmonics, and also to give an elementary interpretation of some known photorefractive phenomena [23–25]. However, the basic theoretical results obtained up to now are valid in an approximation linear in the wave amplitudes. Such a linear theory describes the threshold conditions for the parametric excitation of space-charge waves, and the rate of exponential growth (increment of the instability) of infinitely small wave amplitudes. It cannot describe the final state of the waves beyond the threshold.

The aim of the present paper is to lay the foundation of the nonlinear theory of parametrically excited space-charge waves in sillenites. This includes an analysis of the stabilization mechanism for the parametric instability, a calculation of the amplitudes of the split and unsplit subharmonics, and an investigation of stability of subharmonic regimes against small perturbations.

In Sec. II we introduce nonlinear equations for the space-charge field, and on the basis of these discuss the most important properties of space-charge waves. This section also contains a summary of the main results of the linear theory of the parametric instability in sillenites.

In Sec. III we consider one of the basic concepts of the nonlinear theory, the nonlinear frequency shift for space-charge waves. We distinguish two types of nonlinear shifts, namely, the eigenshift $\delta\omega_{11}$, owing to the action of wave 1 on itself, and the mutual shift $\delta\omega_{12}$, due to the effect of wave 2 on wave 1. It is remarkable that the contributions to $\delta\omega_{12}$ coming from the forced oscillations with spatial frequencies $\vec{k}_1+\vec{k}_2$ and $\vec{k}_1-\vec{k}_2$ partly compensate for each other, and that this partial compensation is preserved even in the limit $\vec{k}_2\to\vec{k}_1$. The latter is connected with the long-range electrostatic interaction. We apply the general expressions for $\delta\omega_{12}$ to the cases of longitudinal and transverse split of the main subharmonic $K/2$.

In Sec. IV we find and investigate the stationary solutions for split and unsplit subharmonics beyond the threshold of parametric instability. These solutions give the amplitudes and phases of the excited waves, which are important for the subsequent analysis of stability of the nonlinear regimes. We show that the nonlinear solutions can be simply interpreted in terms of the nonlinear frequency shifts. The difference of the nonlinear shifts $\delta\omega_{11}$ and $\delta\omega_{12}$ in the limit $\vec{k}_2\to\vec{k}_1$, leads, in particular, to the result that an infinitely small splitting of the subharmonic $\vec{K}/2$ is accompanied by a finite change of the energy of the space-charge field.

In Sec. V we show that the split as well as unsplit subharmonics do more than cause frequency shifts for weak waves; they also considerably renormalize the coupling coefficients of those weak waves with the pump, i.e., with the running light pattern. We find explicit expressions for the renormalized coupling coefficients and discuss them.

In the same Sec. V we use the results of Secs. III and IV to find the region of instability of the main subharmonic $K/2$ against the excitation of small three-dimensional perturbations. Our analysis shows that such an instability region always exists. However, not far beyond the threshold of the subharmonic generation, it is restricted to the vicinity of the point $\vec{K}/2$. In other words, the instability of the main subharmonic may be qualified as a modulation one. One can expect that a small spread of the wave vectors is able to stabilize such an instability. In Sec. VI we summarize the results obtained, and discuss further developments and applications of the theory.

II. BASIC RELATIONS

Our starting point is the following three-dimensional nonlinear equation for the potential of the space charge φ:

$$\Delta\varphi_{zt}-\frac{\omega_0}{l_s}\Delta\varphi-\frac{1}{l_0}\Delta\varphi_t+\omega_0\Delta\varphi_z+\frac{l_D^2}{l_0}\Delta^2\varphi_t$$
$$=-\frac{e}{\overline{\epsilon\epsilon_0}}\frac{\alpha}{\hbar\omega}\delta I_z+\frac{e}{\overline{\epsilon\epsilon_0}}\frac{\alpha}{\hbar\omega}\frac{1}{E_0}\mathrm{div}(\delta I\vec{\nabla}\varphi)$$
$$+\frac{1}{E_0}\mathrm{div}(\Delta\varphi_t\vec{\nabla}\varphi), \qquad (1)$$

where Δ is the Laplace operator. This equation describes charge transfer in a photorefractive sillenite crystal subject to an external electric field E_0 (parallel to the z axis), and to light with intensity $I=I_0+\delta I$, where I_0 and δI are the spatially homogeneous and spatially oscillating parts of I, respectively. For a moving light pattern, created by a pair of laser beams detuned with a frequency Ω, we have

$$\delta I=mI_0\cos(Kz-\Omega t), \qquad (2)$$

where m is the contrast of the interference pattern and K the fundamental grating vector, equal to the difference of the pump wave vectors.

In Eq. (1) we have used the following notation: e is the elementary charge, $\overline{\epsilon\epsilon_0}$ is the dielectric susceptibility, α is the absorption coefficient, $\hbar\omega$ is the energy of a light quantum, $\omega_0=\alpha I_0/N_t\hbar\omega$ is a characteristic frequency, N_t is the effective trap concentration, $l_0=\mu\tau E_0$ is the drift length in the external field, μ is the mobility of the photoelectrons, τ their lifetime, $l_D=\sqrt{k_BT\mu\tau/e}$ is the diffusion length, k_B is the Boltzmann constant, T is the absolute temperature, $l_s=\overline{\epsilon\epsilon_0}E_0/eN_t$ is the characteristic screening length, and the subscripts z and t are the differentiation with respect to the longitudinal coordinate and the time, respectively.

The procedure of deriving Eq. (1) and the region of its applicability were described in detail in our earlier paper [19]. Actually, Eq. (1) describes accurately the propagation, damping, parametric excitation, and nonlinear interaction of space-charge waves in crystals with sufficiently large drift length for moderate light intensity.

The first two terms on the left-hand side of Eq. (1) describe dispersive and lossless wave propagation; if we restrict ourselves to these terms and put $\varphi\propto\exp(i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t-\gamma_{\vec{k}}t)$, we obtain the dispersion law

$$\omega_{\vec{k}}=\frac{eg_0}{\overline{\epsilon\epsilon_0}E_0}\frac{1}{k_z}, \qquad (3)$$

where $g_0=\alpha I_0/\hbar\omega$ is the generation rate of photoelectrons. Such a dependence of the frequency on the wave vector is highly unusual for waves. Actually, Eq. (3) is valid only in a restricted region of the $\vec{k}$ space.

The last three terms on the left-hand side of Eq. (1) characterize the wave damping,

$$\gamma_{\vec{k}} = g_0 \left(\frac{1}{N_t} + \frac{e}{\overline{\epsilon\epsilon_0} \mu \tau E_0^2 k_z^2} + \frac{k_B T}{\overline{\epsilon\epsilon_0} E_0^2} \frac{k^2}{k_z^2} \right), \tag{4}$$

which is clearly positive and even in $\vec{k}$ and $\vec{E}_0$. The first, second, and third terms in the brackets are connected with recombination, drift, and diffusion of photoelectrons, respectively. Only the last (diffusion) term depends on the transverse component of the wave vector $\vec{k}_\perp$; this term grows with $k_\perp^2$.

If the material parameters meet the inequality $eN_t \mu \tau \gg \overline{\epsilon\epsilon_0}$ (which is valid for the sillenites), the condition for the weakness of the wave damping, $\gamma_{\vec{k}} \ll \omega_{\vec{k}}$, is found to be fulfilled for a large region of applied fields and wave vectors [18,19]. Without an applied field the weakly damped space-charge waves are absent.

The three terms on the right-hand side of Eq. (1) describe linear excitation of the space-charge field by a moving light pattern, parametric excitation of the waves, and the effects of the nonlinear wave interaction, respectively. If we discard the last two terms, accept Eq. (2) for δI, and put $\varphi = \varphi_K \exp(iKz - i\Omega t) +$ c.c., for the amplitude of the electrostatic potential φ_K we obtain

$$\varphi_K = -i \, \frac{mE_0}{2K} \frac{\omega_K}{\Omega - \omega_K + i\gamma_K}. \tag{5}$$

For $\Omega \simeq \omega_K$ this formula describes nothing else than the linear resonant excitation of an oscillator with eigenfrequency ω_K and damping γ_K.[1] Far from resonance, $|\Omega - \omega_K| \gg \gamma_K$, the damping γ_K may be neglected.

Spatial subharmonics, split as well as unsplit, are the result of the instability of the moving fundamental grating with respect to the excitation of weakly damped space-charge waves. The wave vectors of the excited waves meet the following well-known conditions of spatially temporal (parametric) resonance:

$$\Omega = \omega_{\vec{k}_1} + \omega_{\vec{k}_2}, \qquad \vec{K} = \vec{k}_1 + \vec{k}_2. \tag{6}$$

They are also called decay conditions [4,9], being associated with the transformation of an oscillation quantum $\vec{K},\Omega$ into a wave pair 1 and 2 [4,9]. In the general case, Eqs. (6) yield a surface in $\vec{k}$ space, the decay surface. Any given vector $\vec{k}_1$ related to this surface is coupled with another vector $\vec{k}_2$, and vice versa. For our particular dispersion law (3), the decay surface is given by the equations

[1]The so-called resonant enhancement of the photorefractive response [23].

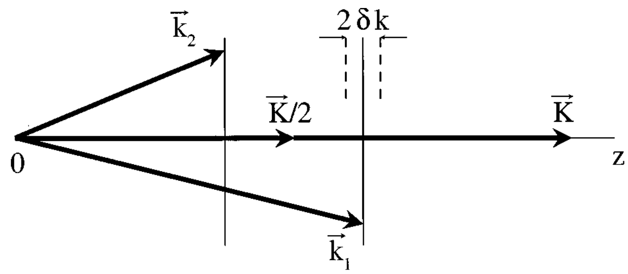


FIG. 1. Geometrical scheme for the decay of the fundamental grating for $\Omega > 4\omega_K$. The vertical solid lines stand for the decay surface; the dotted lines indicate the boundary of the stability region. δk is the width of the decay surface (see the last paragraph of Sec. II). It is only indicated for one sheet of the decay surface.

$$(k_{1,2})_z = \frac{K}{2} \, (1 \pm \sqrt{1 - 4\varepsilon}), \qquad \vec{k}_{1\perp} = -\vec{k}_{2\perp}, \tag{7}$$

where ε is the useful dimensionless parameter

$$\varepsilon = \frac{\omega_K}{\Omega} = \frac{e}{\overline{\epsilon\epsilon_0}} \frac{\alpha I_0}{\hbar \omega} \frac{1}{E_0 K \Omega}. \tag{8}$$

We see that Eqs. (7) can be fulfilled only for $\varepsilon \leqslant 1/4$, that is, for a frequency $\Omega \geqslant 4\omega_K$. For $\Omega > 4\omega_K$ the decay surface consists of a pair of planes perpendicular to the z-axis; see Fig. 1. The corresponding wave vectors $\vec{k}_1$ and $\vec{k}_2$ belong to different planes and have opposite transverse projections. For $\Omega = 4\omega_K$ (i.e., for $\varepsilon = \frac{1}{4}$) the two planes coincide; the special case $\vec{k}_1 = \vec{k}_2 = \vec{K}/2$ corresponds here to the first subharmonic. For $\Omega < 4\omega_K$, the decay is forbidden.

To find the threshold and increment of the decay instability, one has to insert into Eq. (1) the ansatz

$$\varphi = \varphi_K \, e^{i(Kz - \Omega t)} + \varphi_1 \, e^{i(\vec{k}_1 \cdot \vec{x} - \omega_{\vec{k}_1} t)} + \varphi_2 \, e^{i(\vec{k}_2 \cdot \vec{x} - \omega_{\vec{k}_2} t)} + \text{ c.c.}, \tag{9}$$

and to isolate next the terms with spatial frequencies $\vec{k}_1$ and $\vec{k}_2$. As a result, one obtains the following coupled system of linear equations for the slowly varying amplitudes φ_1 and φ_2^*,

$$\left(\frac{d}{dt} + \gamma_{\vec{k}_1} \right) \varphi_1 = iV_1 \varphi_2^*, \qquad \left(\frac{d}{dt} + \gamma_{\vec{k}_2} \right) \varphi_2^* = -iV_2^* \varphi_1. \tag{10}$$

The expression for the coupling coefficients V_1 and V_2 is the same up to an interchange of the indices 1 and 2, and

$$V_1 = \frac{K\Omega}{k_1^2 k_{1z}} \left[\frac{m}{2} \, \varepsilon \, \vec{k}_1 \cdot \vec{k}_2 - e_K \left(\vec{k}_1 \cdot \vec{k}_2 + \varepsilon \, k_2^2 \, \frac{k_{1z}}{k_{2z}} \right) \right]. \tag{11}$$

Here $e_K = -iK\varphi_K E_0$ is the dimensionless amplitude of the space-charge field for the fundamental grating. In the decay region, $4\varepsilon \leqslant 1$, we find from Eq. (5) that

$$e_K \simeq \frac{m}{2} \frac{\varepsilon}{1-\varepsilon}. \tag{12}$$

Since e_K is real, the coupling constants $V_{1,2}$ are also real.

Each coupling constant consists of several contributions. The first term in square brackets in Eq. (11) comes from the second term of the right-hand side of the starting equation (1), and the last two terms, proportional to e_K, originate from the last term of Eq. (1). In the important special case $\varepsilon = \frac{1}{4}$, when $k_{1z} = k_{2z} = K/2$ and $k_1^2 = k_2^2$, from Eq. (11) we obtain

$$V_1 = V_2 = \frac{\Omega}{4(1+\tan^2\theta)}[m - 10e_K + \tan^2\theta(6e_K - m)], \tag{13}$$

where θ is the angle between the vectors $\vec{k}_{1,2}$ and the z axis, see Fig. 1. Terms of different origin (proportional to m and proportional to e_K) are clearly seen in Eq. (13). Obviously a compensation of the first two terms in square brackets leads to a growth of $V_{1,2}$ with increasing θ, whereas a compensation of the last two terms results in an angular decrease of the coupling coefficients $V_{1,2}$. In the framework of the present approximation, linear in $\varphi_{1,2}$, the amplitude e_K is given by Eq. (12); for $\varepsilon = \frac{1}{4}$ it gives $6e_K = m$, i.e., the total compensation of the last two terms in Eq. (13). As we shall see below, Eq. (10) also remains valid beyond the framework of the theory linear in the amplitudes $\varphi_{1,2}$, whereas the expression for e_K alters because of the coupling of the fundamental grating with the excited waves. Note that the contributions to e_K allowing for nonresonant excitation of the higher spatial harmonics, $2K$, $3K$, ... remain negligible for $\varepsilon \lesssim 1/4$.

Putting $\varphi_1, \varphi_2^* \propto \exp(\Gamma t)$ in Eq. (10), for the increment of the instability (characteristic exponent) we obtain

$$\Gamma = -\frac{1}{2}(\gamma_{\vec{k}_1} + \gamma_{\vec{k}_2}) + \sqrt{4(\gamma_{\vec{k}_1} - \gamma_{\vec{k}_2})^2 + \Gamma_0^2}, \tag{14}$$

where $\Gamma_0^2 = V_1 V_2^*$. Since the product $V_1 V_2^*$ is positive, the increment Γ is real. If the difference of the dampings $\gamma_{\vec{k}_{1,2}}$ is relatively small, $|\gamma_{\vec{k}_1} - \gamma_{\vec{k}_2}| \ll \gamma_{\vec{k}_{1,2}} \simeq \gamma$, we have $\Gamma = \Gamma_0 - \gamma$; this assumption is often justified.

Since V_1 and e_K are given by Eqs. (11) and (12), a decrease of the transverse component of the wave vector (equal in value for waves 1 and 2) favors the instability. Putting $k_\perp \equiv (k_{1,2})_\perp = 0$, we find the following formula for the threshold contrast:

$$m_{\text{th}} = \frac{1-\varepsilon}{\varepsilon} \sqrt{\frac{\gamma_{k_1}\gamma_{k_2}}{\omega_{k_1}\omega_{k_2}}}. \tag{15}$$

The right-hand side of this expression is, of course, a function of the frequency Ω. The minimum of m_{th} takes place at $\Omega = 4\ \omega_K$ (and therefore at $\varepsilon = \frac{1}{4}$), which corresponds to the excitation of the first subharmonic, $K/2$. Here we have

$$m_{\text{th}} = 3\ \gamma_{K/2}/\omega_{K/2}. \tag{16}$$

For $\Omega < 4\ \omega_K$ the decay conditions (6) are no longer fulfilled, whereas increasing Ω in comparison with $4\omega_K$ results in increasing m_{th}.

The relations (14)–(16) are obtained for a pair of waves 1 and 2 which meet the resonance conditions (6) exactly. Each resonance has, however, a finite width. For this reason, as a result of the instability, waves are also excited with wave vectors close to the decay surface; see Fig. 1. The physical width of the decay surface can be estimated as $\delta k \approx k\ \Gamma_{\vec{k}}/\omega_{\vec{k}} \ll k$. The larger the increment $\Gamma_{\vec{k}}$, the wider the region of the instability.

III. NONLINEAR FREQUENCY SHIFTS

The concept of the nonlinear frequency shift forms the basis of the nonlinear theory of photorefractive subharmonics. Nonlinear frequency shifts are determined by the eigen nonlinearity of the medium. To describe them we neglect wave damping and put $\delta I = 0$. Equation (1) takes then the form

$$\Delta\left(\varphi_{zt} - \frac{\omega_0}{l_s}\varphi\right) = \frac{1}{E_0}\,\text{div}(\Delta\varphi_t \vec{\nabla}\varphi). \tag{17}$$

Frequency shifts may be of two types, eigenshifts and mutual ones. The eigenshift is created by the wave itself. The eigenshift of the frequency of the wave with frequency $\omega_{\vec{k}_1}$ with the wave vector $\vec{k}_1$ will be denoted by $\delta\omega_{11}$. The frequency shift of wave 1 can also be caused by other waves. The change of the frequency $\omega_{\vec{k}_1}$ by a wave with wave vector $\vec{k}_2$ will be called $\delta\omega_{12}$.

To calculate the eigenshift $\delta\omega_{11}$, we seek the potential, φ, in the form

$$\varphi = \varphi_1\ e^{i(\vec{k}_1\cdot\vec{x} - \omega_{\vec{k}_1}t)} + \varphi_2\ e^{2i(\vec{k}_1\cdot\vec{x} - \omega_{\vec{k}_1}t)} + \text{c.c.}, \tag{18}$$

where $\varphi_{1,2}$ are slowly varying amplitudes. The second term in Eq. (18) allows for the forced oscillation with wave vector $2\vec{k}_1$ and frequency $2\omega_{\vec{k}_1}$. Inserting Eq. (18) into Eq. (17), and separating the spatial harmonics, we find

$$\frac{d\varphi_1}{dt} = -\frac{10\ \omega_1 k_1^2}{k_{1z}\ E_0}\ \varphi_2\varphi_1^*, \qquad \varphi_2 = i\ \frac{k_1^2}{6\ k_{1z}\ E_0}\ \varphi_1^2. \tag{19}$$

We have neglected the temporal derivative of φ_2, taking into account the smallness of the correction $\delta\omega_{11}$ compared to $\omega_1 \equiv \omega_{\vec{k}_1}$. Inserting φ_2 into the equation for φ_1, we obtain the equation for φ_1 in the form $d\varphi_1/dt = -i\,\delta\omega_{11}\varphi_1$ with

$$\frac{\delta\omega_{11}}{\omega_1} = \frac{5\ \ |\vec{e}_1|^2}{3}(1+\tan^2\theta), \tag{20}$$

where $\vec{e}_1 = -i\vec{k}_1\varphi_1 E_0^{-1}$ is the normalized amplitude of the space-charge field for wave 1, and θ is the angle between wave vector $\vec{k}_1$ and the z axis. We see that the eigenshift $\delta\omega_{11}$ is a positive increasing function of $k_\perp$.

To find the mutual shift $\delta\omega_{12}$, which characterizes the effect of wave 2 on wave 1, one has to calculate the forced oscillations with the sum $\vec{k}_1 + \vec{k}_2$ and the difference, $\vec{k}_1 - \vec{k}_2$

of the spatial frequencies. Correspondingly, the mutual shift consists of two terms, $\delta\omega_{12}=\delta\omega_{12}^{+}+\delta\omega_{12}^{-}$. The general expressions for $\delta\omega_{12}^{\pm}$ are derived analogously to the calculation of $\delta\omega_{11}$, and are given by

$$\delta\omega_{12}^{+}=\frac{\vec{k}_1\cdot\vec{k}_2(\vec{k}_1+\vec{k}_2)^2(\omega_1+\omega_2)+k_2^2(\vec{k}_1\cdot\vec{k}_2+k_1^2)\omega_2}{k_{1z}k_1^2k_2^2(\vec{k}_1+\vec{k}_2)^2}\times\frac{k_2^2(\vec{k}_1\cdot\vec{k}_2+k_1^2)\omega_2+k_1^2(\vec{k}_1\cdot\vec{k}_2+k_2^2)\omega_1}{(k_{1z}+k_{2z})(\omega_1+\omega_2)-k_{2z}\;\omega_{\vec{k}_2}}\,|\vec{e}_2|^2, \tag{21}$$

$$\delta\omega_{12}^{-}=\frac{\vec{k}_1\cdot\vec{k}_2(\vec{k}_1-\vec{k}_2)^2(\omega_1-\omega_2)+k_2^2(k_1^2-\vec{k}_1\cdot\vec{k}_2)\omega_2}{k_{1z}k_1^2\;k_2^2(\vec{k}_1-\vec{k}_2)^2}\times\frac{k_1^2(\vec{k}_1\cdot\vec{k}_2-k_2^2)\omega_1-k_2^2(\vec{k}_1\cdot\vec{k}_2-k_1^2)\omega_2}{(k_{1z}-k_{2z})(\omega_1-\omega_2-\omega_{\vec{k}_1-\vec{k}_2})}\,|\vec{e}_2|^2.$$

Here and in the rest of Sec. III we use the abbreviation $\omega_{1,2}=\omega_{\vec{k}_{1,2}}$, as long as it does not lead to misunderstanding.

It is easy to see that the total nonlinear shift of the frequency, $\delta\omega_1$, is given by the sum of all possible elementary (eigen and mutual) shifts.

Two important properties of the contributions $\delta\omega_{12}^{\pm}$ stem from Eqs. (21): (i) In the limit $\vec{k}_2\to\vec{k}_1$ the following relation holds true: $\delta\omega_{12}^{+}=2\,\delta\omega_{11}$. (ii) The contribution $\delta\omega_{12}^{-}$ has no limit for $\vec{k}_2\to\vec{k}_1$, and, for $|\vec{k}_2-\vec{k}_1|\ll k_1$, we have

$$\frac{\delta\omega_{12}^{-}}{\omega_1}=-2|\vec{e}_2|^2\frac{(\vec{q}\cdot\vec{k}_1)^2}{q^2k_{1z}^2}, \tag{22}$$

where $\vec{q}=\vec{k}_1-\vec{k}_2$. Therefore the shift $\delta\omega_{12}^{-}$ is negative for wave vectors $\vec{k}_1$ and $\vec{k}_2$ close to each other, and it depends strongly on the angle between the vectors $\vec{q}$ and $\vec{k}_1$.

The first property is common for nonlinear media in which the four-wave interactions are due to the quadratic nonlinearity. The nonlinear frequency shifts for surface waves [26] and spin waves [8] may serve as examples. The second property is specific for space-charge waves because for most types of waves the contribution $\delta\omega_{12}^{-}$ tends to zero for $\vec{k}_1\to\vec{k}_2$. We stress that this peculiarity is due to the long-range electrostatic interaction; it is not connected with the approximations used.

Below we consider some important special cases. In the one-dimensional case $(k_{1,2})_\perp=0$, the expressions for $\delta\omega_{12}^{\pm}$ are simplified considerably:

$$\frac{\delta\omega_{12}^{\pm}}{\omega_1}=2|e_2|^2f^{\pm}\left(\frac{k_1}{k_2}\right), \tag{23}$$

where

$$f^{\pm}(r)=f^{\pm}(r^{-1})=\frac{r^2\pm 3r+1}{r^2\pm r+1}. \tag{24}$$

For $r=1$ the functions $f^{+}(r)$ and $f^{-}(r)$ have a maximum and a minimum, respectively, with $f^{+}(1)=\frac{5}{3}$ and $f^{-}(1)=-1$. Away from the extremum the function $f^{+}(r)$ decreases monotonically to 1, whereas $f^{-}(r)$ changes its sign and becomes positive; see Fig. 2. For $k_1=k_2$ we find for the total mutual shift

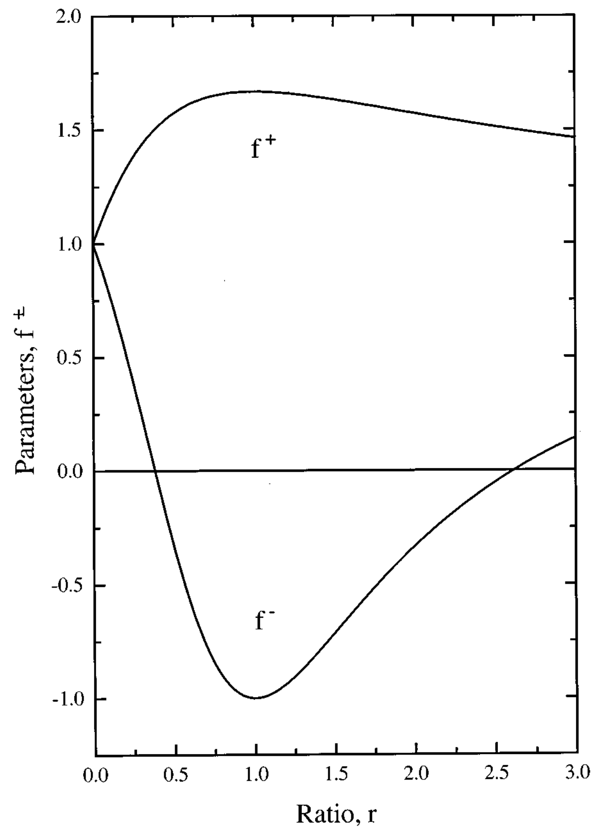


FIG. 2. Dependence of the parameters $f^{\pm}$ on the ratio of the wave vectors r.

$$\frac{\delta\omega_{12}}{\omega_1}=\frac{4}{3}\,|e_2|^2\quad . \tag{25}$$

Let us consider now a pair of two-dimensional configurations (a) and (b), shown in Figs. 3(a) and 3(b). In these cases we have $k_{1z}=k_{2z}$, i.e., the unperturbed frequencies of waves 1 and 2 are the same. For small propagation angles, $\tan^2\theta\ll 1$, we obtain, from Eqs. (21),

$$\frac{\delta\omega_{12}}{\omega_1}=\frac{10}{3}\,|\vec{e}_2|^2(1-\tfrac{1}{5}\tan^2\theta), \tag{26a}$$

$$\frac{\delta\omega_{12}}{\omega_1}=\frac{10}{3}\,|\vec{e}_2|^2(1-\tfrac{8}{5}\tan^2\theta), \tag{26b}$$

respectively, for the configurations (a) and (b). The contribution $\delta\omega_{12}^{-}$ is here negligible. The case of large angles θ is of little interest because of the large damping of the waves.

We note that a correction $\delta\omega_{12}$ to the eigenfrequency $\omega_1=\omega_{\vec{k}_1}$ can be given not only by an eigenmode 2 but also by a forced oscillation with spatial frequency $\vec{k}_2$ and temporal frequency $\omega_2\neq\omega_{\vec{k}_2}$. In particular, the fundamental grat-

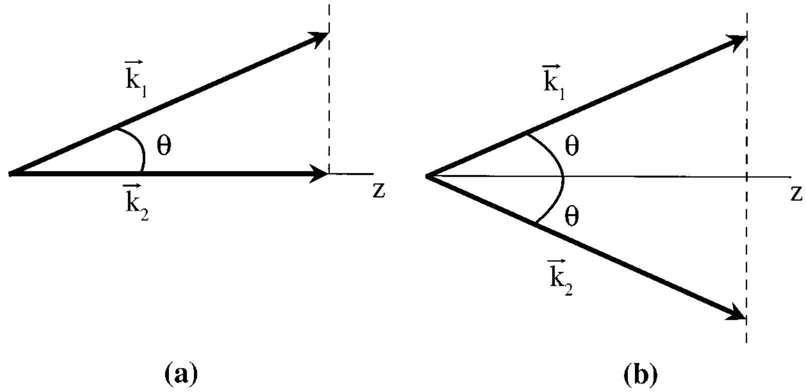


FIG. 3. Basic two-dimensional configurations for the mutual frequency shift.

ing, induced by a running light pattern, may play the role of such an oscillation. In similar cases, the frequency ω_2 in Eqs. (21) is just the frequency of forced oscillations.

Note finally that the frequency shifts introduced above are very convenient to use. They account for certain nonlinear contributions of the second-order perturbation theory by means of a simple renormalization of the coefficients of the linear theory. Furthermore, it is known [27,28] that the nonlinear frequency shifts play an important role in the stabilization of the parametric instability.

IV. STATIONARY SOLUTIONS FOR SUBHARMONICS

A. Unsplit subharmonic *K*/2

Let us put the following into in Eq. (1):

$$\varphi=\varphi_K\, e^{i(Kz-\Omega t)}+\varphi_{K/2}\, e^{\,i/2\,(Kz-\Omega t)}+\text{c.c.} \tag{27}$$

We suppose that the amplitude $\varphi_{K/2}$ is a finite quantity and the frequency Ω is near its limiting value $4\omega_K$. Separating the spatial frequencies $K/2$ and K and introducing a dimensionless amplitude of the space-charge field, $e_k=-ik\varphi_k E_0^{-1}$, with k equal to K or $K/2$, we obtain the following coupled system of nonlinear equations:

$$(\gamma_{K/2}-i\,\delta_{K/2})e_{K/2}=-\frac{i}{2}\,\omega_{K/2}(m-10\,e_K)e^*_{K/2},$$
$$e_K=\frac{m}{6}-\frac{1}{3}\,e^2_{K/2}, \tag{28}$$

where $\delta_{K/2}=0.5\,\Omega-\omega_{K/2}$ is the linear detuning. The second equation describes the effect of the subharmonic on the fundamental grating.

Inserting the second of Eqs. (28) into the first one, we obtain the following remarkable equation for the amplitude $e_{K/2}$:

$$e_{K/2}(\gamma_{K/2}-i\,\delta_{K/2}+\tfrac{5}{3}i\,\omega_{K/2}\,|e_{K/2}|^2)=i\,\frac{m}{3}\,\omega_{K/2}\,e^*_{K/2}. \tag{29}$$

This shows clearly the mechanism of stabilization of the parametric instability for increasing subharmonic amplitude, namely the mismatch of the parametric resonance owing to the frequency shift $\delta\omega_{11}$; see Eq. (20). Actually, Eq. (29) is the threshold condition for the instability, allowing for the renormalization of the frequency. We emphasize that this renormalization comes from the nonlinear correction to $e_K(e_{K/2})$ in Eq. (28).

Writing $e^2_{K/2}=|e_{K/2}|^2\exp(i\Phi)$, from Eq. (29), we easily obtain

$$|e_{K/2}|^2=\tfrac{1}{5}(3\Delta+\sqrt{m^2-m^2_{th}}),$$
$$\sin\Phi=\frac{m_{th}}{m}, \tag{30}$$

where the threshold value of the contrast m_{th} is given by Eq. (16), and

$$\Delta=\frac{\delta_{K/2}}{\omega_{K/2}}=1-4\varepsilon. \tag{31}$$

The dimensionless parameter Δ [a stand-alone symbol; not the Laplace operator as in Eq. (1)] is simply the normalized linear detuning, which is small against 1. This solution for the subharmonic $K/2$ holds true only above threshold, $m>m_{\text{th}}$. The value $|e_{K/2}|^2$ follows a square-root law with increasing contrast m, linearly with increasing normalized detuning Δ. For $\Delta\approx1$, when $|e_{K/2}|$ becomes comparable with 1, our approximation loses its applicability. Near threshold the phase Φ is about $\pi/2$, and far away from threshold, $m\gg m_{\text{th}}$, it is close to zero. As we shall see in Sec. VI, knowledge of the phase Φ is necessary for analyzing the stability of the steady state.

Within the calculation procedure used we may not require smallness of $|e_{K/2}|$ and of the correction to e_K, given by Eq. (28), in comparison with $|e_K|$. The relatively large value of $|e_{K/2}|$ is connected with the fact that the subharmonic is caused by a resonant parametric process, whereas the fundamental grating is a forced oscillation far away from the linear resonance, $\Omega-\omega_K\simeq3\,\omega_K\gg\gamma_K$.

In accordance with Eqs. (28) and (31) the excitation of the subharmonic gives rise to an imaginary part of the amplitude of the fundamental grating e_K, that is, to a spatial shift between the fundamental grating and the interference light pattern. As is known, such a shift is of importance for the optical photorefractive phenomena [29]. In particular, it should result in an energy exchange between the pump beams which form the running light pattern.

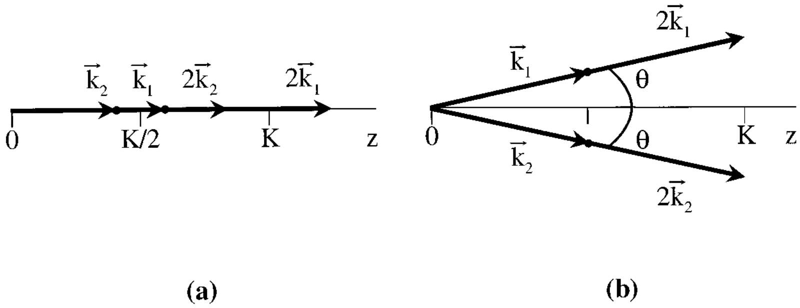


FIG. 4. Geometrical schemes clarifying the longitudinally (a) and transversally (b) split subharmonics. Only the vectors $\vec{k}_1$ and $\vec{k}_2$ and their multiples are indicated by arrows. The arrows of $\vec{K}/2$ and $\vec{K}$ are omitted, but their lengths are indicated.

Above we took into consideration only the spatial frequencies K, $K/2$, and the corresponding eigenshift $\delta\omega_{11}$ for the space-charge wave with wave vector $k_1 = K/2$. Including into the theory the mutual shift for the subharmonic $\delta\omega_{12}$ caused by the fundamental grating (i.e., the terms with the spatial frequency $3K/2$) would result in corrections quadratic in m. One can show that these corrections remain small up to $m = 1$. Likewise one can understand that nonlinear corrections to the damping $\gamma_{K/2}$ would lead to additional terms in Eqs. (29) and (30) which are of order $\gamma_{K/2}^2/\omega_{K/2}^2$.

B. Split subharmonics

For one and the same value of the frequency Ω we can find (in addition to the above solution for the main subharmonic $K/2$) a whole family of solutions for split subharmonics. A split subharmonic consists of a pair of waves with wave vectors $\vec{k}_1$ and $\vec{k}_2$ and frequencies ω_1 and ω_2, the sum of which is $\vec{K}$ and Ω, respectively. In general, the frequencies $\omega_{1,2}$ are different from the eigenfrequencies $\omega_{\vec{k}_{1,2}}$ because of the nonlinear shifts. The equations for the wave amplitudes φ_1 and φ_2, replacing Eq. (29), are

$$[\gamma_1 + i(\delta\omega_1 - \delta_1)]\varphi_1 = iV_1\varphi_2^*,$$

$$[\gamma_2 - i(\delta\omega_2 - \delta_2)]\varphi_2^* = -iV_2^*\varphi_1. \tag{32}$$

Here $\gamma_{1,2} = \gamma_{\vec{k}_{1,2}}$ are the dampings of the waves, $V_{1,2}$ the coupling coefficients defined by Eqs. (11) and (12), $\delta_{1,2} = \omega_{1,2} - \omega_{\vec{k}_{1,2}}$ the linear detunings, and $\delta\omega_{1,2}$ the overall nonlinear frequency shifts of waves 1 and 2 including the appropriate eigenshifts and mutual contributions.

Equations (32) furnish an opportunity to clarify the basic difference between split and unsplit subharmonics. For a split subharmonic, apart from the eigenshifts related to the spatial frequencies $2\vec{k}_1$ and $2\vec{k}_2$, there is a mutual shift, corresponding to the spatial frequencies, $\vec{k}_1 \pm \vec{k}_2$. Because of the difference between eigenshifts and mutual frequency shifts, (see Sec. III), the values of $\delta\omega_{1,2}$ in Eqs. (32) do not tend to $\delta\omega_{11} = 5\ \omega_{K/2}\,|e_{K/2}|^2/3$ for $\vec{k}_{1,2} \to \vec{K}/2$. In other words, even an infinitely small split of the subharmonic $K/2$ gives a finite change of the energy of the space-charge field including the energy of the fundamental grating.

The absence of a continuous transition between the split and unsplit subharmonics is connected with the breaking of the spatial symmetry by the splitting. The unsplit subharmonic corresponds to a periodic solution for the space-charge field (with a period $4\pi/K$); any infinitely small split breaks this periodicity.

In the following calculations we restrict ourselves to small splits, assuming that $|2\vec{k}_{1,2} - \vec{K}| \ll K$ and $|2\omega_{1,2} - \Omega| \ll \Omega$. Furthermore, to avoid clumsy expressions, we consider only the types of splits simplest in symmetry, namely longitudinal and transverse splits; see Fig. 4.

The longitudinal split [see Fig. 4(a)] corresponds to a one-dimensional problem. The wave vectors $k_{1,2}$ may here be presented in the form

$$k_{1,2} = \frac{K}{2}(1 \pm \kappa). \tag{33}$$

The small dimensionless parameter κ characterizes the degree of the split. At first we assume the linear detunings $\delta_{1,2}$ in Eq. (32) to be the same, which corresponds to equal intensities of the waves 1 and 2. In this case,

$$\delta_{1,2} = \omega_{K/2}(\Delta - \kappa^2), \tag{34}$$

where Δ is given again by Eq. (31). For the overall nonlinear frequency shifts $\delta\omega_1$, $\delta\omega_2$ we obtain, using the results of Sec. III,

$$\delta\omega_{1,2} = \frac{\omega_{K/2}}{3}(5|e_{1,2}|^2 + 4|e_{2,1}|^2), \tag{35}$$

where the first term corresponds to the eigenshift and the second one to the mutual shift. The dampings $\gamma_{1,2}$ and the coupling constants $V_{1,2}$ [see Eq. (13)] are $\gamma_{K/2}$ and $-m\ \omega_{K/2}/3$, respectively, in the leading approximation in κ. Taking into account Eqs. (34) and (35), from Eq. (32) we obtain

$$|e_{1,2}|^2 = \tfrac{1}{3}(\Delta - \kappa^2 + \tfrac{1}{3}\sqrt{m^2 - m_{\text{th}}^2}). \tag{36}$$

The case $\Delta = \kappa^2$ corresponds to the excitation of waves exactly meeting the decay conditions $\omega_{1,2} = \omega_{k_{1,2}}$. As follows from Eq. (36), an increase of the split leads to a decrease of the intensities $|e_{1,2}|^2$.

A more general solution for $e_{1,2}$ may be obtained for unequal (but not too different) linear detunings δ_1 and δ_2. This gives different intensities $|e_1|^2$ and $|e_2|^2$. The sum of the intensities, $|e_1|^2 + |e_1|^2$, remains, however, the same as for the case $\delta_1 = \delta_2$.

The sum of the intensities of the split components, $|e_1|^2 + |e_2|^2$, does not tend to $|e_{K/2}|^2$ when the relative value of the split κ goes to zero. As follows from Eqs. (30) and (36), independent of the value Δ we have

$$(|e_1|^2+|e_2|^2)_{\kappa=0}=\tfrac{10}{9}|e_{K/2}|^2 \quad . \tag{37}$$

It is not difficult to find from Eqs. (32) that the sum of the phases of the parametric waves, $\Phi=\arg(e_1 e_2)$, is again given by Eq. (30).

Let us now consider the symmetric transverse split; see Fig. 4(b). In this case the linear detunings are the same, $\delta_{1,2}=\omega_{K/2}\,\Delta$. Since $|\vec{e}_1|^2=|\vec{e}_2|^2$, the nonlinear shifts $\delta\omega_1$ and $\delta\omega_2$ are equal. To find them we use Eqs. (20) and (26b). In the leading approximation in θ^2, we obtain

$$\delta\omega_{1,2}=5\,\omega_{K/2}\,|\vec{e}_{1,2}|^2(1-\tfrac{11}{15}\theta^2)\,. \tag{38}$$

Note that in this case there is no mutual contributions to the overal nonlinear frequency shifts $\delta\omega_1$ and $\delta\omega_2$. As follows from Eq. (13), both coupling constants V_1 and V_2, are, with the same accuracy, equal to $-m\;\omega_{K/2}(1-\theta^2)/3$. It is worthwhile to remind the reader that the effect of waves 1 and 2 on the amplitude e_K is equivalent to the already considered renormalization of the frequencies. For this reason, we use Eq. (12) of the linear theory for e_K. As for the wave dampings $\gamma_{1,2}$, we choose them equal to $\gamma_{K/2}$, neglecting a possible angular dependence. The point is that the first two terms (which do not depend on θ) usually dominate in Eq. (5) for $\gamma_{\vec{k}}$ so that the dependence $\gamma(\theta)$ is relatively weak. Taking this into account, from Eq. (32) we obtain

$$|\vec{e}_{1,2}|^2\simeq\tfrac{1}{15}(1+\tfrac{11}{15}\theta^2)[3\Delta+\sqrt{m^2(1-\theta^2)^2-m_{\rm th}^2}\,]\,. \tag{39}$$

The angular dependence under the radical describes an increase of the threshold of the instability owing to a decrease of the coupling constants.

In the limit $\theta\to 0$, we have

$$(|\vec{e}_1|^2+|\vec{e}_2|^2)_{\theta=0}=\tfrac{2}{3}|e_{K/2}|^2\,. \tag{40}$$

In such a way, an infinitely small transverse split leads to a decrease of the subharmonic energy of about 30%. The reason for this conspicuous drop is the absence of the negative contribution $\delta\omega_{12}^-$ to the mutual frequency shift. For the transverse split the sum of the phases of waves 1 and 2 is identical to that for the previous cases.

V. ANALYSIS OF STABILITY

One mechanism which may cause the instability of the subharmonics against small perturbations is quite clear. This is again the decay of the fundamental grating in accordance with the resonance conditions (6). The wave vectors $\vec{k}_{1,2}$ refer here to a pair of weak waves, and the fundamental grating is depleted by the coupling with the subharmonic. This depletion alters, in particular, the relation between two different contributions to the coupling coefficients $V_{1,2}$; see Eq. (11). The weak waves 1 and 2 experience nonlinear frequency shifts, caused mainly by the subharmonic. The shifts coming from the fundamental grating are of minor importance, for they are nearly the same for the strong and weak space-charge waves with the wave vectors $\vec{k}_{1,2}\simeq\vec{K}/2$.

Apart from the above mechanism affecting the instability, there is one more important factor. It is related to another nonlinear process, namely, to the conversion of two subharmonic wave quanta into a pair of weak waves, $\vec{k}_{1,2}$. Clearly, this process meets again the decay conditions (6). We shall see below that this additional process results in renormalization of the coupling coefficients $V_{1,2}$ given by Eq. (11).

In accordance with the aforesaid we can represent the small perturbation of the potential $\delta\varphi$ in the form

$$\delta\varphi=a_1\,e^{i(\vec{k}_1\cdot\vec{x}-\omega_1 t)}+a_2\,e^{i(\vec{k}_2\cdot\vec{x}-\omega_2 t)}+\text{c.c.}, \tag{41}$$

where the frequencies $\omega_{1,2}$ are near the eigenvalues $\omega_{\vec{k}_{1,2}}$. Below we derive equations for the amplitudes $a_{1,2}$ to analyze the renormalization both of the frequencies and of the coupling coefficients. For the sake of simplicity we restrict ourselves to the case of the main subharmonic $\vec{K}/2$.

A. Renormalization of the coupling coefficients

Figure 5 shows the wave vectors $\vec{k}_{1,2}$ meeting the condition $\vec{k}_1+\vec{k}_2=\vec{K}$. The difference gratings, created by the subharmonic $\vec{K}/2$ together with the waves 1 and 2, respectively, have the same spatial frequency, $\vec{q}$:

$$\vec{q}=\vec{k}_1-\vec{K}/2=\vec{K}/2-\vec{k}_2\,. \tag{42}$$

This means that the gratings in question are responsible not only for the contributions $\delta\omega^-$ to the mutual frequency shifts for waves 1 and 2 (see Sec. III), but also for the mutual coupling of these waves. This coupling is different from the one considered above (in Sec. III) because it is due to the finite amplitude of the subharmonic.

To describe this additional coupling one should return to Eq. (17), which incorporates all actual linear and nonlinear terms. First of all, we calculate the amplitude of the forced oscillation of the potential, $\varphi_{\vec{q}}$. Taking into account Eq. (41), for $q\ll K/2$ we have

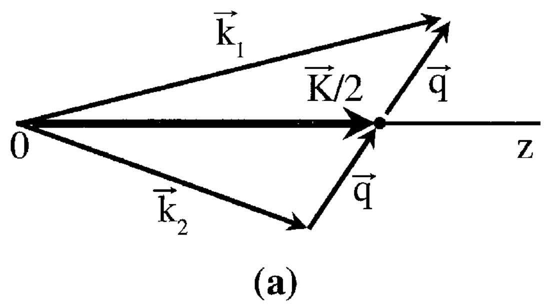
$$\varphi_{\vec{q}}=\frac{(\theta^2+2\kappa)a_1 e_{K/2}^*+(\theta^2-2\kappa)a_2^* e_{K/2}}{\theta^2+\kappa^2}, \tag{43}$$

where θ is again the angle between the vectors $\vec{k}_{1,2}$ and the z axis (see Fig. 5) and $\kappa=2q_z/K$. The components $(k_{1,2})_z$ are expressed in terms of κ, using Eqs. (33). From Eq. (43) one can see that the amplitude $\varphi_{\vec{q}}$ depends strongly on the orientation of the vector $\vec{q}$.

Next we obtain the equations for the slowly varying amplitudes $a_{1,2}$, describing the diffraction of the subharmonic $K/2$ from the grating $\vec{q}$. Using Eq. (17) we obtain

$$\frac{da_1}{dt}=i\,\omega_{K/2}\,\frac{\theta^2+\kappa}{\theta^2+1}\,\varphi_{\vec{q}}\;e_{K/2}\,,$$
$$\frac{da_2^*}{dt}=-i\,\omega_{K/2}\,\frac{\theta^2-\kappa}{\theta^2+1}\,\varphi_{\vec{q}}\;e_{K/2}^*\,. \tag{44}$$

Inserting Eq. (43) into Eqs. (44), we arrive at the following matrix equation for a_1 and a_2^*:



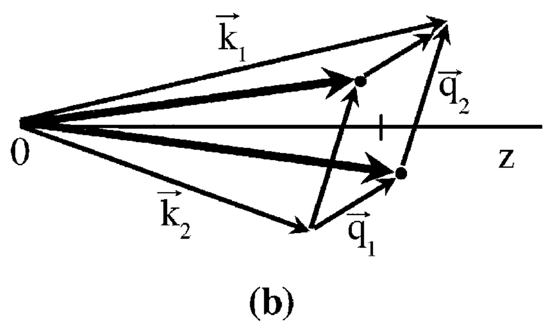


FIG. 5. Geometrical schemes for the renormalization of the coupling constants of the weak parametric waves 1 und 2 by unsplit (a) and split (b) subharmonics. Thick vectors refer to strong waves. The sum of the wave vectors $\vec{k}_1$ and $\vec{k}_2$ is equal to the fundamental grating vector $\vec{K}$.

$$\frac{d}{dt}\begin{pmatrix} a_1 \\ a_2^* \end{pmatrix} = \begin{pmatrix} -i\,\delta\omega_{12}^- & i\,\delta V_1 \\ -i\,\delta V_2^* & i\,\delta\omega_{21}^- \end{pmatrix}\begin{pmatrix} a_1 \\ a_2^* \end{pmatrix}. \tag{45}$$

The diagonal matrix elements again describe the mutual frequency shifts, whereas the off-diagonal elements give the nonlinear corrections to the coupling coefficients V_1 and V_2^*. The expressions for $\delta\omega^-$ and δV are most important in the region $\theta \lesssim |\kappa| \ll 1$, where they are not negligibly small. In this region we have

$$\delta\omega_{12}^- = \delta\omega_{21}^- = -2\,\omega_{K/2}\,|e_{K/2}|^2\left(1+\frac{\theta^2}{\kappa^2}\right)^{-1},$$
$$\delta V_1 = \delta V_2 = -2\;\omega_{K/2}\;e_{K/2}^2\left(1+\frac{\theta^2}{\kappa^2}\right)^{-1}. \tag{46}$$

The expressions for the frequency shifts fully agree with Eq. (22). The correction to the coupling coefficients is generally complex; it depends not only on $|e_{K/2}|^2$ but also on the phase $\Phi = \arg(e_{K/2})^2$. The largest correction takes place in the one-dimensional case; with an increasing value of the transverse component of the wave vectors, $k_\perp$, it decreases rapidly.

The renormalized coupling coefficients for waves 1 and 2 are obviously $\tilde{V}_{1,2} = V_{1,2} + \delta V_{1,2}$, with $V_{1,2}$ and e_K given by Eqs. (11) and (12), respectively. For $\kappa \ll 1$ and $\theta^2 \ll 1$ we obtain

$$V_1 \simeq V_2 \simeq -\frac{\omega_{K/2}}{3(1+\theta^2)}(m - 5e_{K/2}^2 + 3\theta^2 e_{K/2}^2),$$
$$\tilde{V}_1 \simeq \tilde{V}_2 \simeq -\frac{\omega_{K/2}}{3(1+\theta^2)}(m - ce_{K/2}^2 + 3\theta^2 e_{K/2}^2), \tag{47}$$

where $c = 5 - 6(1+\theta^2/\kappa^2)^{-1}$. We see that the renormalization results in a negative contribution to the parameter c; in the one-dimensional case this term changes the sign of c. It should be understood that keeping θ^2 in the angular factor $(1+\theta^2)^{-1}$ gives a more accurate result than necessary for the region $\theta \lesssim |\kappa|$, where $\delta V_{1,2}$ is comparable with $V_{1,2}$. Outside this region we have $\delta V_{1,2} \ll V_{1,2}$ and $\tilde{V}_{1,2} \simeq V_{1,2}$. Let us also to note that the inequality $\theta \lesssim \kappa \ll 1$, defining the region of strong renormalization, is equivalent to the condition $|q_\perp| \lesssim |q_z| \ll 1$. The geometric meaning of this condition is clear from Fig. 5. Finally, we remind the reader that $e_{K/2}^2$ is given by Eqs. (30).

B. Threshold equation

To find the threshold conditions for the instability of the subharmonic $K/2$, we have to use the following equations, incorporating all the above-considered causes for changing the amplitudes of small perturbations, $a_{1,2}$:

$$\left[\frac{d}{dt} + \gamma_1 + i(\delta\omega_1 - \delta_1)\right]a_1 = i\tilde{V}_1 a_2^*,$$
$$\left[\frac{d}{dt} + \gamma_2 - i(\delta\omega_2 - \delta_2)\right]a_2^* = -i\tilde{V}_2 a_1\quad. \tag{48}$$

The parameters $\delta_{1,2} = \omega_{1,2} - \omega_{\vec{k}_{1,2}}$ are again the linear detunings. They can compensate for the nonlinear frequency shifts and as such favor the instability. The overall nonlinear frequency shifts for the weak waves, $\delta\omega_1$ and $\delta\omega_2$, are the mutual shifts induced by the subharmonic $K/2$. They are given by the appropriate formulas of Sec. III. The renormalized coupling coefficients $\tilde{V}_{1,2}$ are specified by Eq. (47).

Let us next put in Eqs. (48) $a_1, a_2^* \propto \exp(i\Gamma''t)$, where Γ'' is an unknown real parameter. It can be considered as the imaginary part of the increment Γ, the real part of which is zero. Actually, we use the general form of the solution for $a_{1,2}$, allowing for temporal oscillations at the threshold of the instability. Multiplying two algebraic equations [which result from Eqs. (48)], and canceling the common factor $a_1 a_2^*$, we obtain the following real threshold equation:

$$\gamma^2 + (\delta\overline{\omega} - \overline{\delta})^2 = R + (I/2\gamma)^2, \tag{49}$$

where

$$\delta\overline{\omega} = (\delta\omega_1 + \delta\omega_2)/2, \quad R = \mathrm{Re}(\tilde{V}_1\tilde{V}_2^*),$$
$$\overline{\delta} = (\delta_1 + \delta_2)/2, \quad I = \mathrm{Im}(\tilde{V}_1\tilde{V}_2^*), \tag{50}$$

and $\gamma = \gamma_{K/2}$. When deriving Eq. (49) we again neglected the difference between $\gamma_{1,2}$ and $\gamma_{K/2}$. A more detailed analysis shows that this difference leads to no real effect, but only to clumsy formulas. One can also show that the subharmonic is unstable when the right-hand side of Eq. (49) exceeds the left-hand one.

We should now find explicit expressions for the parameters given by Eqs. (50). Accepting representation (33) for the projections $(k_{1,2})_z$ we obtain that the average linear detuning $\delta \simeq \omega_{K/2}(\Delta - \kappa^2)$. The dimensionless parameter Δ retains its former meaning, and is given by Eq. (31).

For the average nonlinear shift, $\delta\overline{\omega}$, we obtain, using Eqs. (22), (26a), (30), and (46),

$$\delta\overline{\omega} \simeq 2\,\omega_{K/2}|e_{K/2}|^2\left(\frac{5}{3} - \frac{\kappa^2}{\kappa^2+\theta^2}\right). \tag{51}$$

In deriving Eq. (51) we neglected the weak angular dependence of $\delta\omega_{12}^{+}$ in comparison with the strong dependence of $\delta\omega_{12}^{-}$; the latter is expressed by the negative contribution to $\delta\overline{\omega}$. This approximation is fully justified for the following analysis. Furthermore, using Eqs. (30) and (47), we find, for $\theta \lesssim |\kappa|$,

$$R \simeq \gamma_{K/2}^2 + \omega_{K/2}^2\left[\frac{1}{3}\,|e_{K/2}|^2\left(5 + \frac{\kappa^2 - 5\,\theta^2}{\kappa^2+\theta^2}\right) - \Delta\right]^2 - \omega_{K/2}^2|e_{K/2}|^4\frac{\kappa^2\ \theta^4}{(\kappa^2+\theta^2)^2},$$

$$\frac{I^2}{4\,\gamma_{K/2}^2} \simeq \omega_{K/2}^2|e_{K/2}|^4\frac{\kappa^2\ \theta^4}{(\kappa^2+\theta^2)^2}. \tag{52}$$

As follows from Eqs. (52), the last contribution to R (which is small against the first two terms for $\theta \lesssim |\kappa|$) compensates for the term $(I/2\gamma)^2$ in the threshold equation (49).

C. Instability of the subharmonic *K*/2

We begin our analysis of the threshold equation with the one-dimensional case. Putting $\theta = 0$ in Eqs. (51) and (52), from Eq. (49) we obtain the following simple condition for instability,

$$\kappa^2 < \tfrac{2}{3}\,|e_{K/2}|^2. \tag{53}$$

This condition shows that the subharmonic $K/2$ is always unstable. The instability is of the modulational type, because it results in the excitation of waves with the wave vectors near to $K/2$. The smaller the amplitude of the subharmonic, the narrower the region of instability.

Let us compare the limiting value of the parameter κ^2 corresponding to the instability $\kappa_l^2 = 2\,|e_{K/2}|^2/3$, with the value $\kappa_0^2 = \Delta$ meeting the decay conditions (7). Using Eq. (30), we find

$$\frac{\kappa_l^2}{\kappa_0^2} = \frac{2}{5}\left(1 + \frac{\sqrt{m^2 - m_{\text{th}}^2}}{3\Delta}\right). \tag{54}$$

We see that the relative width of the instability region becomes smaller with increasing $\Delta = 1 - 4\varepsilon$, and tends to 2/5.

It is worthy of note that the instability of the subharmonic $K/2$ originates from the difference gratings with wave vectors $\vec{k}_{1,2} - \vec{K}/2$, and more precisely from the negative contribution to the nonlinear frequency shift $\delta\omega_{12}^{-}$. In the absence of such a contribution the subharmonic would be stable, because the positive contribution to the nonlinear frequency shift, $\delta\omega_{12}^{+} \simeq 10\ \omega_{K/2}\,|e_{K/2}|^2/3$, would shift weak waves out of the parametric resonance for arbitrary values of $|e_{K/2}|^2$ and κ. This fact may be checked by dropping the terms in the threshold equation which stem from the difference gratings.

We have seen earlier [see Eqs. (22) and (46)], that the difference contributions to the nonlinear frequency shifts and to the coupling constants $\widetilde{V}_{1,2}$ decrease rapidly with increasing angle θ between the wave vectors of the weak waves and the z axis. Therefore one can expect that the region of instability is narrow not only with respect to κ but also with respect to θ. In line with this assumption, from Eqs. (49)–(52) one can obtain the following inequality restricting the instability region in the $\vec{q}$ plane:

$$\frac{\theta^2}{\kappa^2} < \frac{2|e_{K/2}|^2 - 3\,\kappa^2}{10|e_{K/2}|^2 + 3\,\kappa^2}. \tag{55}$$

This condition, as well as inequality (53), does not explicitly contain the parameter $\Delta = 1 - 4\varepsilon$. As follows from Eq. (55), the maximum possible value of θ is not larger than $0.17\,|e_{K/2}|$, which is much smaller than 1. This value corresponds to $\kappa \simeq 0.55\,|e_{K/2}|$. Note that the ratio θ/κ is nothing else but the tangent of the angle of inclination of the vector $\vec{q}$ against the z axis; see Fig. 5. The value of this inclination angle becomes larger with decreasing κ, but it does not exceed 11.3°. Figure 6 shows the region of instability for several representative values of $|e_{K/2}|$.

Note finally that terms of higher order in θ^2 (which are not important in the above considered angular region, $\theta \lesssim |\kappa|$) act stabilizingly on the weak waves propagating under large angles, $\theta \gg |\kappa|$. Therefore the above region of instability is unique for $|e_{K/2}|^2 \ll 1$.

VI. DISCUSSION

We have considered in detail the main elements of the nonlinear theory of space-charge waves. This theory enables one to describe the parametric excitation of these waves by a running light pattern, to find various stationary subharmonic regimes beyond the threshold of the instability, and to analyze the stability of these nonlinear regimes against small perturbations. One of the basic elements of the theory are the nonlinear eigenshift and mutual frequency shift. These shifts strongly affect the above-threshold behavior of the parametric waves, bringing them in or removing them from resonance. In particular, the amplitudes of the split and unsplit subharmonics are determined by the appropriate nonlinear frequency shifts. The analysis of the stability of the stationary solutions involves not only the frequency shifts of weak waves but also other elements of the theory: the renormalization of the coupling coefficients and the feedback of the subharmonics on the pump. Here not only the intensities of the parametrically excited waves but also their phases are of importance. Some important details of the analysis are connected with the peculiar features of the system considered, in particular with the unusual dispersion law of the space-charge waves and with the presence of the anomalously large ‘‘difference’’ contributions to the frequency shifts.

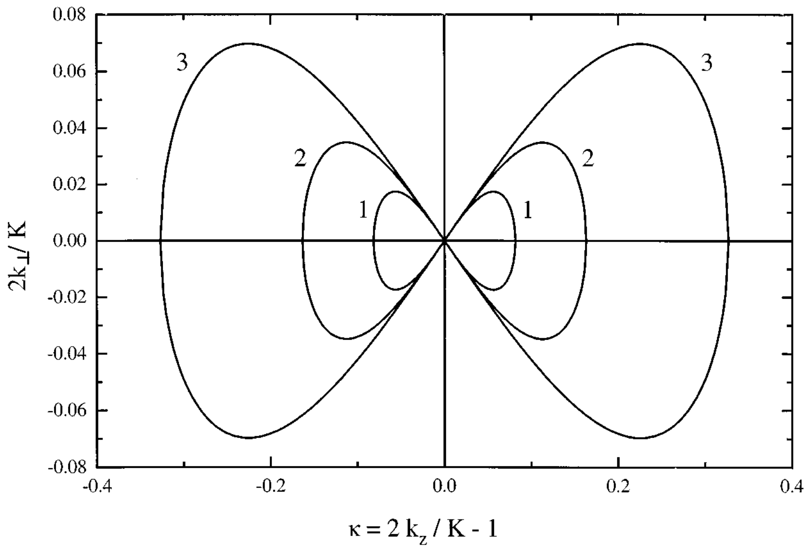


FIG. 6. Boundary of the instability region of the subharmonic $K/2$ for several values of $|e_{K/2}|$. Curves 1, 2, and 3 correspond to $|e_{K/2}|^2 = 0.1$, 0.2, and 0.4, respectively.

Under the same external conditions (the same fundamental grating vector $\vec{K}$, frequency Ω, contrast m, and intensity I_0) the nonlinear equations admit a whole family of stationary solutions for split and unsplit subharmonics. These solutions differ considerably in the energy of the space-charge field. We have shown that the main subharmonic $K/2$ is modulationally unstable. This gives some hope that the instability may be stopped by a small broadening of the Fourier spectrum of the excited waves.

An exhaustive analysis of stability of various nonlinear regimes is beyond the scope of our paper, although such a study should not present fundamental difficulties. In particular, it is easy to understand that the main modification of the theory necessary for analyzing the stability of the split subharmonics is taking into account not only one spatial grating $\vec{q}$ for each pair of weak waves, but two gratings, $\vec{q}_1$ and $\vec{q}_2$, see Fig. 5(b). The structure of Eq. (48) remains the same in this case.

Among the subharmonics considered in Sec. IV, the transversally split one has the greatest chance to be stable. The point is that the negative contribution $\delta\omega_{12}^-$ is negligibly small for any weak wave pair, with the wave vectors near to the primary ones $\vec{k}_1$ and $\vec{k}_2$; see Fig. 4(b). The positive frequency shift $\delta\omega_{12}^+$ is not small here, and it represents an important stabilizing factor for the instability.

It should be noted that the transverse split may occur in an arbitrary plane including the fundamental grating vector; see Fig. 4. This axial degeneration enables one to construct not only bichromatic solutions for the split subharmonics but also more complicated steady states which include an infinite set of parametric wave pairs. Such wave regimes should be described not dynamically but statistically [9].

The possibilities of a numerical simulation of the initial dynamic equations for the space-charge field are, probably, restricted to the one-dimensional case. It would be expedient to verify numerically the fact of the modulation instability of the main sumharmonic $K/2$ in this case, and to find the final wave distribution beyond the threshold of the parametric instability.

For future theoretical studies of the subharmonics careful experiments are of upmost importance. Unfortunately, the majority of experiments performed up to now has only demonstrated one or another instability of the fundamental grating. A detailed experimental study of the subharmonics requires, first of all, a high homogeneity of the pump intensity I_0 and of the external field E_0. An inhomogeneity of these parameters results in a spatial modulation of the frequency ω_K and, therefore, in the coexistence of different nonlinear regimes in different parts of the sample. Under these circumstances optical measurements can only inform about average properties of the parametrically excited space-charge waves. We hope that the development of a nonlinear theory of space-charge waves and the performance of careful experiments will furnish a better understanding of space-charge wave instabilities and the effect of the generation of subharmonics on the photorefractive properties of sillenites.

ACKNOWLEDGMENTS

This paper was supported in part by INTAS (International Organization for the Promotion of Co-operation with Scientists from the Independent States of the Former Soviet Union), Grant No. 94-248.

[1] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Course of Theoretical Physics (Pergamon, Oxford, 1986), Vol. 5.
[2] D. D. Joseph, *Stability of Fluid Motion*, Springer Tracts in Natural Philosophy Vol. 27 (Springer-Verlag, Berlin, 1976).
[3] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Plenum, New York, 1984), Vol. 1.
[4] V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I: Wave Turbulence*, Springer Series in Nonlinear Dynamics (Springer-Verlag, Berlin, 1992).
[5] H. M. Gibbs, *Optical Bistability* (Academic, London, 1985).

[6] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley-Interscience, New York, 1984).

[7] *Nonlinear Dynamics and Tubulence*, edited by G. I. Barenblatt, G. Iooss, and D. D. Joseph (Pitman, Boston, 1983).

[8] V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, Phys. Rep. **129**, 285 (1985).

[9] V. S. L'vov, *Wave Turbulence Under Parametric Excitation. Application to Magnets*, Springer Series in Nonlinear Dynamics (Springer-Verlag, Berlin, 1994).

[10] R. F. Kazarinov, R. A. Suris, and B. I. Fuks, Fiz. Tekh. Poluprovodn. **7**, 688 (1973) [Sov. Phys. Semicond. **7**, 480 (1973)].

[11] S. Mallick *et al.*, J. Appl. Phys. **63**, 5660 (1988).

[12] D. J. Webb and L. Solymar, Opt. Commun. **74**, 386 (1990).

[13] J. Richter, A. Grunnet-Jepsen, J. Takacs, and L. Solymar, IEEE J. Quantum Electron. **30**, 1645 (1994).

[14] J. Takacs, M. Schaub, and L. Solymar, Opt. Commun. **91**, 252 (1992).

[15] J. Takacs and L. Solymar, Opt. Lett. **17**, 247 (1992).

[16] J. Takacs and L. Solymar (unpublished).

[17] H. Pedersen and P. M. Johansen, Opt. Lett. **19**, 1418 (1994).

[18] B. I. Sturman, M. Mann, and K. H. Ringhofer, Appl. Phys. A **55**, 235 (1992).

[19] B. I. Sturman, M. Mann, J. Otten, and K. H. Ringhofer, J. Opt. Soc. Am. B **10**, 1919 (1993).

[20] B. I. Sturman, M. Mann, and K. H. Ringhofer, Opt. Lett. **17**, 1620 (1992).

[21] T. E. McClelland, D. J. Webb, B. I. Sturman, and K. H. Ringhofer, Phys. Rev. Lett. **73**, 3082 (1994).

[22] B. I. Sturman *et al.*, J. Opt. Soc. Am. B **12**, 1621 (1995).

[23] P. Refregier, L. Solymar, H. Rajbenbach, and J.-P. Huignard, J. Appl. Phys. **58**, 45 (1985).

[24] S. I. Stepanov and M. P. Petrov, Opt. Commun. **53**, 292 (1985).

[25] T. E. McClelland *et al.*, Opt. Commun. **113**, 371 (1995).

[26] V. E. Zakharov, Prikl. Mekh. Tekh. Fiz. **2**, 86 (1968).

[27] L. D. Landau and E. M. Lifshitz, *Mechanics*, Course of Theoretical Physics (Pergamon, Oxford, 1969), Vol. 1.

[28] C. Hayashi, *Nonlinear Oscillations in Physical Systems* (Princeton University Press, Princeton, 1985).

[29] *Photorefractive Materials and Their Applications I and II*, edited by P. Günter and J.-P. Huignard, Topics in Applied Physics Vols. 61 and 62 (Springer-Verlag, Berlin, 1988).